Local scaling of the flux for standardlike maps

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Results of systematic numerical explorations of local scaling of the flux near a critical invariant circle for a class of standardlike maps are reported. The scaling law is universal and the universality class is determined apparently only by the tail in the rotation number of the critical invariant circle. The flux near just created cantori is also studied, and another type of scaling is indicated, which is a combination of the scaling laws in the critical and the strongly supercritical regimes.

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I. INTRODUCTION

The main impediment to a successful development of a theory of transport in the phase space of a typical Hamiltonian system is the existence of complicated structures on all scales [1–8]. In two extreme cases of an integrable and a fully chaotic systems the descriptions of transport are relatively simple. In the first case, the dynamics is a simple combination, fixed by the distribution of the initial points, of ergodic systems on the corresponding invariant tori of half the dimension of the phase space. In the second case, an initial distribution is spread by a diffusion process all over the entire energy surface, and the standard theory of diffusion could be successfully applied [9,10]. Near-integrable systems, or systems on a part of the phase space almost filled by the invariant KAM tori, can be treated by perturbation theories of the adiabatic invariants. The transport in systems close to a fully chaotic one is also well described by neglecting small domains with regular dynamics and with any near-critical structures such as the reminiscent of the KAM tori. The later, usually called cantori [11], or the Aubry-Mather sets [12,13], are known to represent very strong but partial barriers for the transport [14].

Most of the essential features of the problem are present already in systems with only two degrees of freedom, and the associated maps on the sections of the surfaces of constant energy, i.e., one-parameter families of area-preserving maps. The parameter is interpreted as related to the energy of the system, and usually describes the deviations from an integrable linear map. Genericity conditions require the maps to be smooth diffeomorphisms, and it is physically plausible to consider only the maps for which the action variable is proportional to the difference between the successive iterates of the angle coordinate. The maps that satisfy the stated conditions are known as the twist area-preserving diffeomorphisms of a cylinder, and are the systems for which the problems of transport have been studied most extensively [3]. The central quantity in such theories is the flux of points from certain noninvariant domains in one iteration of the map. In particular, domains bounded by the so-called rotational curves, i.e., those that go around the cylinder, and can be thought of as graphs of functions, i.e., motion along the curve preserves orientation, are analyzed.

As was pointed out, the main difficulty is to describe the transport near and across the strong partial barriers, that is across just created rotational cantori, and thus local properties of the flux near the critical tori and just created cantori are the most important. Besides the rotational cantori, which represent the main partial barriers for the transport along the cylinder, the structure of islands around elliptic periodic points of increasing order with their sticky boundaries are also responsible for the anomalous properties of the diffusion, and have to be taken into account in a more complete theory of transport [15].

All regular (periodic and quasiperiodic) rotational orbits of such diffeomorphisms of the cylinder are enumerated by a single quantity, the corresponding rotation number or the frequency, and the flux through rotational curves also can be considered as a function of it. Thus, one is led to consider a one-parameter family of functional relations between the flux and the rotation number. The dependence on the parameter is smooth; but the dependence on the rotation number, for a fixed nonzero value of the parameter, is very complicated.

Important quantities for typical Hamiltonian systems display fractal dependence on the frequency with evidence of global and local self-similarity. Examples are the critical functions [16–18], critical invariant circles [19,20], boundary circles [21], and actions of minimizing orbits [22]. Theoretical explanation of the local scaling properties of such self-similar fractal objects is provided by the renormalization theory for the area-preserving twist maps [23].

Also based on the renormalization theory is the idea that the local scaling properties should depend only on the asymptotic tail in the continued fraction expansion of the frequency, and that these properties should be universal for a class of the twist maps. For example, critical invariant circles of a large class of area-preserving twist maps have the fractal spectra that depend only on the tail of the continued fraction expansion of the rotation number of the torus, and are independent of the rotation number and the particular form of the map [20].

Although, the existence and universality of a local simple scaling should not, in the view of the renormalization theory, come as a surprise, the important fractal quantities with relatively simple local scaling, the corresponding scaling relations, and the corresponding universality classes still remain to be investigated. A local scaling law near a type of critical invariant circles with the rotation numbers given by the so-called noble irrationals was obtained numerically for the standard map, and justified by the renormalization analyzes (see the following section). The analogous formula has been conjectured for the flux near the critical invariant circles of a more general type, and is expected to be universal, but we are not aware of any reported numerical confirmation.

In this paper we report the results of systematic numerical investigations of the universal scaling properties of the flux near a critical invariant torus with a quadratic irrational rotation number for a class of area-preserving twist maps. We have then studied the variations of the scaling relations in a neighborhood of a weakly supercritical cantorus.

II. DEFINITIONS

The class of twist maps of the cylinder to be considered in this paper is defined by the following equations:

$$p_{n+1}=p_n+\frac{k}{2\pi}f(q_n),\quad q_{n+1}=(q_n+p_{n+1})\quad \mathrm{mod}\quad 1, \tag{1}$$

where $(q,p)\in S^1\times R$ and the function f is a smooth, even or odd, periodic function with a zero average $\int_0^1 f(q)=0$, represented by a finite trigonometric polynomial. In particular, the standard map of Taylor and Chirikov is obtained from Eq. (1) when $f(q)\equiv f_1(q)=\sin(2\pi q)$, and the maps of the class (1) are often called generalized standard maps. The periodicity of the function f is an arbitrary integer.

Regular, i.e., periodic and quasiperiodic, orbits of a map of the form (1) are characterized by their rotation number ρ, or frequency, that is defined as follows:

$$\rho(q_0,p_0)=\lim_{i\to\infty}\frac{x_i}{i}\equiv\lim_{i\to\infty}\frac{\pi_1\overline{T^i}(x_0,p_0)}{i}, \tag{2}$$

where $x_0\equiv q_0$ and $x_i\in R$ are the first coordinates of the successive iterates of the point (x_0,p_0) by the lift $\overline{T}$ of the map (1) on $R\times R$, and π_1 denotes the projection on the first coordinate. If the point (q_0,p_0) belongs to a periodic or a quasiperiodic orbit, then the quantity (2) is well defined and is characteristic of the orbit. Rational ρ correspond to the periodic and irrational ρ to the quasiperiodic orbits.

The flux from any bounded domain in the phase space is defined as the total area of points that leave the domain in one iteration of the map. The flux across a rotational circle on the cylinder is defined as the flux from the semi-infinite domain bounded by the rotational circle, and is specially important for the transport theory. Typically there is upward and downward flux across such circles, but they are the same for maps of the form (1). If the rotational circle is an invariant circle the flux across it is obviously zero. Thus, the KAM tori represent complete barriers to the transport along the p axes. Furthermore, the flux across broken KAM tori, the so-called cantori, is known to be very small for the values of the parameter that are not much larger than the corresponding critical value at which the invariant circle is destroyed. Thus, good understanding and efficient computational procedures for the flux across slightly supercritical cantori are crucial for the transport theory.

We shall now very briefly recapitulate the basic elements of the theory that are needed for an explicit computation of the flux across rotational circles. The maps of the form (1) are canonical, i.e., globally (or exactly) symplectic transformations of the phase space, and, as such, are given by the corresponding generating function $S(x_i,x_{i+1})$, $(x_i,x_{i+1})\in R\times R$, defined by

$$S(x_i,x_{i+1})=(x_i-x_{i+1})^2/2+\frac{k}{4\pi^2}V(x_i),\quad f(x)=-\frac{dV(x)}{dx}, \tag{3}$$

and the map is then given by

$$p_i=-\frac{\partial S}{\partial x_i},\quad p_{i+1}=\frac{\partial S}{\partial x_{i+1}}. \tag{4}$$

The generating function is used to define action functionals on sequences of real numbers, which are then used to calculate the areas between rotational circles and ultimately the flux. In particular, periodic orbits of the period n are given as stationary points of the following action functional on the finite sequences of n real numbers $x_i, i=0,\ldots,n-1$ subject to the condition that $x_0+m=x_n$, for some integer m:

$$A(m/n,k)(x_0,x_1,\ldots,x_{n-1})=\sum_{i=0}^{i=n-1}S(x_i,x_{i+1})|_{x_n=x_0+m}. \tag{5}$$

There are, at least, one minimum and minimax stationary points of Eq. (5) for each m/n where (m,n) are coprime. These give minimizing and minimax periodic orbits of the map (1), with the rotation numbers equal to m/n. Finally the main object of interest in this paper, that is, the flux through a rotational curve constructed by the stable and unstable manifolds of the minimizing (m/n) periodic orbit is given by the difference of the corresponding minimax and minimum of the periodic action (5) [2,3]

$$A_{minmax}(m/n,k)-A_{min}(m/n,k)\coloneqq F(m/n,k). \tag{6}$$

The continued fraction expansion (CFE) of an irrational $\rho=[0,a_1,a_2,\ldots,a_j,\ldots]$ encodes the rotation numbers of the minimizing periodic orbits $m_j/n_j=[0,a_1,a_2,\ldots,a_j]$ that give the best successive approximations (of a given length $n\leqslant n_j$) of the quasiperiodic orbit. For a fixed k the limit of the flux $F(m_l/n_l,k)$ over such a sequence of minimizing periodic orbits $m_l/n_l\to\rho$ exists and is non-negative [13,12]. Depending on k the limit could be zero and then there is an invariant circle supporting the quasiperiodic orbits with the rotation number ρ. Otherwise, if the limit is nonzero the quasiperiodic orbit with the rotation number ρ is supported on a fractal invariant set, the cantorus. The smallest value of the parameter k at which the quasiperiodic orbit is not supported on a smooth circle is called critical and is denoted by $K(\rho)$. The flux through a cantorus is nonzero, and can be defined as the limiting value of the sequence $F(m_j/n_j,k)$ for the fixed value of $k>K(\rho)$, which is such that the quasiperiodic orbit with the rotation number $\rho\leftarrow m_j/n_j$ is on the cantorus.

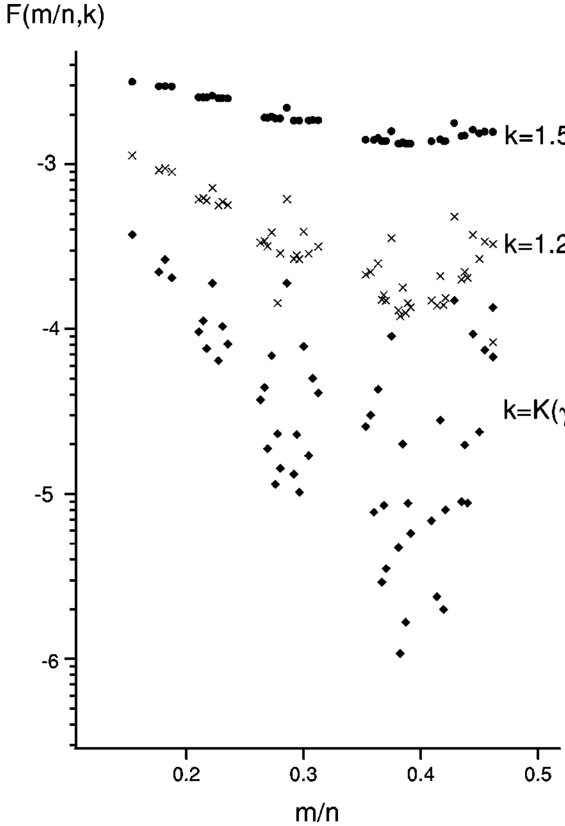

FIG. 1. The flux $F(m/n,k)$ for the first few levels of the Farey tree and few values of the parameter k: $k=K(\gamma)$ (diamonds), $k=1.2K(\gamma)$ (crosses), and $k=1.5K(\gamma)$ (circles). Log scale is used for y axes, where the numbers indicate the powers of 10.

For a quasiperiodic orbit on a circle or on a cantorus, the limit $\kappa=\lim_{n\to\infty}n^{-1}\ln\|DT^n\|$ where $\|DT^n\|$ is the norm of the product of the derivatives of the map along the orbit, exists and is the same for almost any point with respect to a unique ergodic invariant measure, generated by the quasiperiodic orbit. The limit gives the Lyapunov exponent of the invariant circle, which is equal to zero, or the cantorus, when $\kappa\geqslant 0$. In typical cases, the Lyapunov exponent can be estimated from the behavior of the residues of the minimizing periodic orbits with the rotation numbers $m_l/n_l\to\rho$ using the fact that $n_l^{-1}\ln R_{n_l}\to\kappa$ [24].

In general, the flux $F(m/n,k)$ is a complicated function of m/n for any fixed $k>0$ and a smooth increasing function of k for any fixed m/n. Some of its properties are indicated in Fig. 1, which shows $F(m/n,k)$ for the standard map and a set of rationals that belong to the first few levels of the Farey tree, and for various values of the parameter k. The smallest k in Fig. 1 is equal to the critical value $K(\gamma)=0.971\,635\ldots$ at which the last rotational invariant torus is destroyed. $F(m/n,K(\gamma))$ clearly displays the fractal discontinuous structure with large relative variations. Such large relative variations of the flux are present also in the weakly supercritical case, or near just created cantori. For k sufficiently large, most of the local structure becomes negligible and the relative variations of the flux are very small. Similar qualitative picture is obtained for other maps of the type (1). Good understanding of the critical and weakly supercritical behavior of the flux is obviously most difficult.

III. LOCAL SCALING OF THE FLUX: COMPUTATIONS AND RESULTS

Scaling relations of the flux have been studied before, mostly using the standard map as an example, while some sort of universality is expected due to the universal properties of critical invariant circles suggested by the renormalization ideas. For example, it is known that the flux $F(m/n,k)$ near the critical invariant circle with $\rho\equiv\gamma=(\sqrt{5}-1)/2$, i.e., in the limit over CFE approximates of the golden circle for the standard map satisfies the following simple scaling relation [2,3]:

$$F(m_j/n_j,K(\rho))=C_\rho\beta^{-j}\quad\text{for}\quad\rho=\gamma, \tag{7}$$

and it has been conjectured that the same scaling, with the same β, is asymptotically valid for any noble critical circle of the standard map, that is, a critical circle with the CFE of the corresponding ρ that ends with a tail of an infinite sequence of units. Renormalization ideas suggest that the scaling should depend only on the tail for arbitrary tails and furthermore should be universal for a class of the twist area-preserving maps. Furthermore, a more general scaling relation

$$F(m_j/n_j,K(\rho))=Bn_j^{-\alpha} \tag{8}$$

has been suggested in Ref. [21] for any irrational ρ with a periodic tail of its CFE, and numerically tested for the standard map. In the case of irrationals with the constant tail, the two scaling relations (7) and (8) should be equivalent since then $n_j\approx\rho^j$.

Much faster convergence to the limiting value of the flux through a cantorus has been observed for strongly supercritical noble cantori of the standard map. In this case, the following formula has been suggested [2,3]:

$$F(m_j/n_j,k\gg K(\rho))=F_0(\rho)+C\lambda^{-n_j},\quad m_j/n_j\to\rho, \tag{9}$$

where F_0 is the flux through the cantorus, and λ appears to be equal to the Lyapunov exponent of the cantorus. Thus, in the strongly supercritical case the convergence is faster than exponential.

We have undertaken a systematic numerical check of the relations (7)–(9) for various ρ with constant and periodic tails in the CFE and for various maps in the class (1). The weakly supercritical case, i.e., k slightly above $K(\rho)$, is also studied.

The numerical procedure consists basically in calculations of sufficiently long periodic orbits. These are used first to estimate the critical values $K(\rho)$, using the Greene criterion [25], and then for calculations of the actions and the flux via Eq. (5). Variations of the critical values of the parameter $K(\rho)$ of the order 10^{-6} induce variations of the numerical values of the parameters β or α of the order 10^{-3}. Calculation of long periodic orbits near a critical circle or a cantorus represents a nontrivial numerical task. Due to dynamical in-

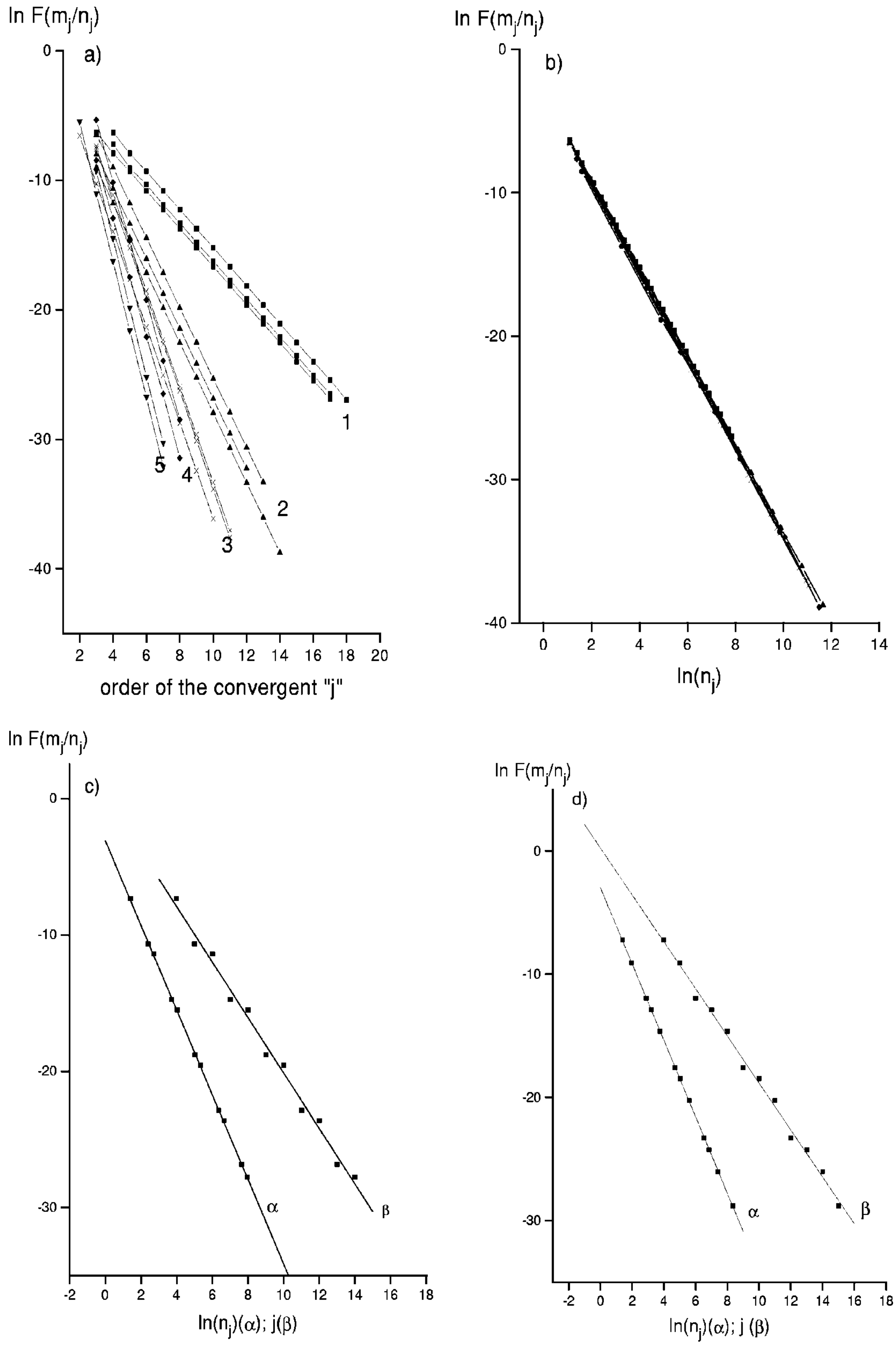

FIG. 2. Scaling relations (7) and (8) for the standard map: Shown in (a) and (b) are examples with constant tails equal to 1^∞ (boxes), 2^∞ (crosses), 3^∞ (diamonds), 4^∞ triangles, and 5^∞ (circles). (c),(d) illustrate the scaling for the rotation numbers with periodic tails $(1,2)^\infty$ (c) and $(1,2,1)^\infty$ (d). The parameter k on each line is equal to the corresponding $K(\rho_i)$.

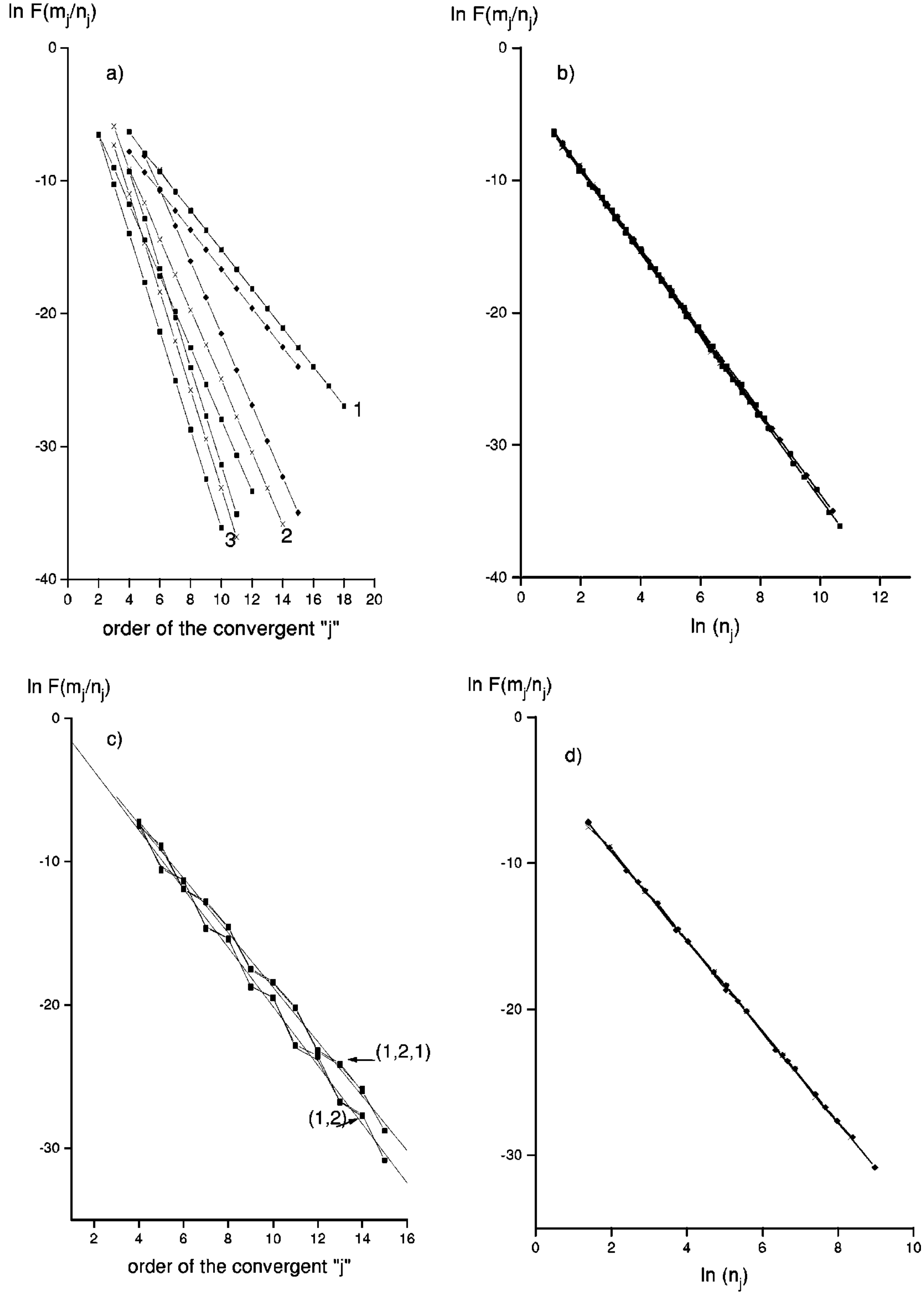

FIG. 3. Universality of the scaling relations (7) (a),(c) and (8) (b),(d) for three examples of the maps (1) with $2f(x)=\sin(2\pi x)$;$f(x)=\sin(2\pi x)/2+\sin(4\pi x)/4$ (diamonds), $3f(x)=\sin(2\pi x)/2+\sin(4\pi x)/4+\sin(6\pi x)/6$ (crosses), and the standard map (boxes). Constant or periodic tails of the CFE are indicated on the figures.

stability of such orbits, an initial error could easily lead to the convergence of the numerical results to a nearby orbit with a similar rotation number that is stable for the considered value of the parameter. This is specially true for the critical invariant circles or cantori with non-noble rotation numbers, because apparently there are always smooth invariant circles with a noble number and the corresponding stable long periodic orbits nearby. We have used two independent procedures for such calculations. The first one represents a numerically stable method to implement a generalization of the Newton method to find the extremal points of the periodic action (4) [26]. Thus, the initial guess and the outcome

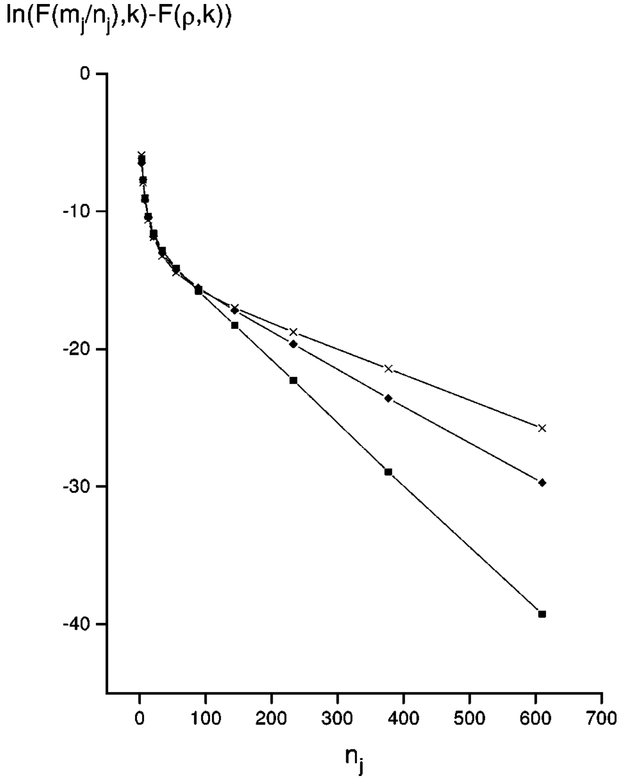

FIG. 4. Illustration of two types of scaling laws in the weakly supercritical regime: Shown are differences $F(m_j/n_j,k)-F(\rho_i,k)$ for the golden tori $\rho_i, i=1,2,3$ of the maps as in Fig. 3, each for the corresponding $k=1.04K(\rho_i)$.

is a set of n points that approximate the whole periodic orbit. The other procedure is based on a generalization of the bisection method to find zeros of polynomials with large order [27]. Efficiency and accuracy of both methods depend crucially on the special techniques to determine a good initial guess for the Newton algorithm, or a good choice of the initial characteristic polyhedra in the bisection method.

The results of our computations are summarized and illustrated in the sequence of Figs. 2–4 and in Tables I–III. We first present the results in the critical case and for the standard map.

Figures 2(a) and 2(b) represent the logarithm of the flux $\ln F(m_j/n_j,K(\rho))$ for the standard map as a function of the order j [Fig. 2(a)] and the denominator n_j [Fig. 2(b)] of the convergent $m_j/n_j\rightarrow\rho$ for the irrational rotation numbers ρ with constant tails equal to $(1)^\infty,(2)^\infty,(3)^\infty,(4)^\infty$, and $(5)^\infty$, where $()^\infty$ denotes that the tail consists of an infinite repetition of $()$. The corresponding data are collected in Table I.

For any critical invariant circle with the rotation number with the constant tail the scaling is of the form (7) or (8) for the corresponding sequence of periodic orbits with rotation numbers $m_j/n_j\rightarrow\rho$. Numerical values of the parameter β or α, for the rotation numbers with the same tail, could be made the same by small variations of the critical values of the perturbation parameter $K(\rho)$ of the order 10^{-6}. On the other hand, β or α are clearly different for the rotation numbers with different tails. Notice that the values of α are similar for all critical circles that we have tested, i.e., those with coefficients of the CFE $a_i\leqslant 5$. The dependence on the tail of the CFE of the scaling properties of the flux is more easily detected in terms of the parameter β of the scaling relation (7). Both parameters increase with the (constant) tail l.

Figures 2(c) and 2(d) represent $\ln F(m_j/n_j,K(\rho))$ for the standard map as a function of the order j and the denominator n_j of the convergent $m_j/n_j\rightarrow\rho$ for the irrational rotation numbers ρ with periodic tails $(1,2)^\infty$ [Fig. 2(c)] and $(1,2,1)^\infty$ [Fig. 2(d)]. The scaling (8) is still valid, and critical circles with rotation numbers with the same tail appear to have the same α, and different tails imply different α. On the other hand the scaling relation (7) is no more valid. However, there are subsequences of the convergents $(m_j/n_j)^s$, $J\rightarrow\infty$, such that the flux over each of the sequences verifies the scaling (7) with the same β. The number of such subsequences is equal to the length of the repeating string of the CFE. So, again a unique value of β can be associated with the periodic tail.

TABLE I. Values of the coefficients β and α in Eqs. (7) and (8) for the standard map and critical invariant tori with rotation numbers as indicated.

ρ	β	α
$[0,1^\infty]$	−1.469	−3.050
$[0,1,2,1^\infty]$	−1.471	−3.048
$[0,3,1^\infty]$	−1.468	−3.048
$[0,2^\infty]$	−2.699	−3.064
$[0,1,2^\infty]$	−2.696	−3.063
$[0,3,2^\infty]$	−2.700	−3.064
$[0,3^\infty]$	−3.694	−3.091
$[0,3,1,3^\infty]$	−3.713	−3.089
$[0,1,3^\infty]$	−3.676	−3.094
$[1,4^\infty]$	4.589	3.182
$[1,1,4^\infty]$	4.607	3.188
$[1,5^\infty]$	5.273	3.216
$[1,1,5^\infty]$	5.308	3.207
$[0,(1,2)^\infty]$	−2.038	−3.101
$[0,(1,2,1)^\infty]$	−1.870	−3.10

TABLE II. Values of the coefficients β and α in Eqs. (7) and (8) for the maps (1) with $2f(x)=\sin(2\pi x)$; $f(x)=\sin(2\pi x)/2+\sin(4\pi x)/4$, $3f(x)=\sin(2\pi x)/2+\sin(4\pi x)/4+\sin(6\pi x)/6$, and critical invariant tori with rotation numbers as indicated.

Tail	$f(x)$	β	α
1^∞	$f_2(x)$	−1.469	−3.055
1^∞	$f_3(x)$	−1.468	−3.052
2^∞	$f_2(x)$	−2.697	−3.062
2^∞	$f_3(x)$	−2.691	−3.060
3^∞	$f_2(x)$	−3.692	−3.092
3^∞	$f_3(x)$	−3.694	−3.091
$(1,2)^\infty$	$f_2(x)$	−2.042	−3.101
$(1,2)^\infty$	$f_3(x)$	−2.046	−3.105
$(1,2,1)^\infty$	$f_2(x)$	−1.875	−3.093
$(1,2,1)^\infty$	$f_3(x)$	−1.871	−3.090

TABLE III. Values of the coefficient λ in Eq. (9), which are equal, up to the given precision, with the Lyapunov exponents of the cantori, for the standard map and maps as in Table II and for few cantori with rotation numbers as indicated, all for the same supercritical $k=1.2$.

ρ	f	λ
$[0,1^\infty]$	f_1	0.234
$[0,3,1^\infty]$	f_1	0.313
$[0,2^\infty]$	f_1	0.246
$[0,1,2^\infty]$	f_1	0.302
$[0,3^\infty]$	f_1	0.311
$[0,1,3^\infty]$	f_1	0.391
$[0,1^\infty]$	f_2	0.395
$[0,3,1^\infty]$	f_2	0.345
$[0,2^\infty]$	f_2	0.455
$[0,1,2^\infty]$	f_2	0.349
$[0,3^\infty]$	f_2	0.360
$[0,1,3^\infty]$	f_2	0.360
$[0,1^\infty]$	f_3	0.493
$[0,3,1^\infty]$	f_3	0.438
$[0,2^\infty]$	f_3	0.449
$[0,1,2^\infty]$	f_3	0.4798
$[0,3^\infty]$	f_3	0.517
$[0,1,3^\infty]$	f_3	0.270

To test the universality of the scalings (7) and (8) in the critical case we computed the flux, for orbits as in Fig. 3, but for several maps of the form (1). In discussions of the universal properties of one-dimensional maps, important role is played by the degree of inflection of the map. It is also known that the qualitative behavior of the maps of the form (1) depends, to a certain extent, on this quantity. For example, if the degree of inflection is higher than three, an invariant circle can be broken and created several times as the parameter k is increased. If the degree is three, as for the standard map, the phenomenon of reoccurrence of an invariant circle is not possible. On the other hand the degree of inflection larger than three does not seem to play important role when fractal properties of the critical circles of the smooth maps (1) are considered [20,28].

Figures 3 illustrates the scaling for the maps with $2f_2(x)=\sin(2\pi x)/2+\sin(4\pi x)/4$ and $3f_3(x)=\sin(2\pi x)/2+\sin(4\pi x)/4+\sin(6\pi x)/6$ compared with the standard map, and for few frequencies with the same tail and for different tails $[(1)^\infty,(2)^\infty,(3)^\infty,(1,2)^\infty,(1,2,1)^\infty]$ The critical values of the parameter k and the global shape of the critical function $K(\rho)$ are different from the case of the standard map but the scaling is qualitatively the same and within the chosen accuracy. The data are given in Table II.

The scaling is of the form (8), for all rotation numbers that we have tested, and α does not depend on the map, but again only on the tail of the rotation number. The scaling (7) is also valid, in the same generalized sense for the periodic tails, as for the standard map. We have tested maps with functions f that have different slopes at zero, different orders of the first derivative of f at zero that is equal to zero (even or odd derivatives depending on the symmetry of the map), different signs of the derivatives of f, etc., but we have not found that any of these qualitative properties of the function f plays any role in the type of the scaling law nor in the actual values of the scaling constants. The periodicity of the map does not play the role, since a different period would only imply a modular transformation of the rotation numbers, which leaves invariant the tail of the CFE. The flux near critical circles for all maps of the form (1) had the same scaling, with the same scaling constants that depend only on the tail of the CFE. These results confirm the expectations based on the renormalization theory.

Let us now discuss the scaling of $F(m/n,k)$ in the case of supercritical k. The results of our computations are illustrated in Fig. 4. In the weakly supercritical case the flux $F(m_j/n_j)$ over the sequence of convergents is characterized by two different types of scalings. The initial part of the sequence of differences $F(m_j/n_j)-F(\rho,k)$, though not very close to $\rho \leftarrow m_j/n_j$ still satisfies the relation (7) or (8), with different exponents. The exponential scaling with the order of the convergents breaks down close to the cantorus with the rotation number ρ and is replaced by the scaling relation (9). The convergence to the limiting value of the flux through the cantorus becomes faster. As the cantorus becomes strongly supercritical the scaling of the flux according to Eq. (9) becomes valid for periodic orbits further away from the cantorus. The values of the flux for the periodic orbits with rotation numbers given by the first few rationals on the Farey tree are sufficient for calculation of the flux everywhere.

The transition through different types of the scaling is qualitatively the same for all maps of the class (1) that we have considered. However, we do not find any quantitative universality. Values of the scaling constants depend on the map, and within one map on the pair $(k>K(\rho),\rho)$. However, the coefficient λ in Eq. (9) is numerically equal to the slope of the linear relation between $\ln R(m_j/n_j)$ and n_j in all the cases that we have calculated. This supports the claim that λ is equal to the Lyapunov exponent of the cantorus. The results are illustrated in Table III, where we give values of λ for a sample of maps and for cantori with different rotation numbers, all for $k=1.2$. Up to the presented accuracy, the coefficient λ, as given by the calculations of the flux and the formula (9), and the slope of $\ln R(m_j/n_j)$ vs n_j i.e., the Lyapunov exponent of the cantorus, are the same. More research is needed to see whether there is any relation between the values of the flux $F(m/n,k)$ between different pairs $(m/n,k)$.

IV. CONCLUSIONS

We have reported results of systematic numerical computation of the flux for a class of twist area-preserving maps of the cylinder. On the basis of the numerical evidence, we can formulate a conjecture about the universal scaling of the flux as follows: Consider a sequence of the periodic orbits of a map of the form (1), which approximate a critical invariant circle with the rotation number ρ and have rational rotation numbers equal to the continued fraction approximates

$m_j/n_j \to \rho$. Then, for any ρ with a constant or periodic tail in the CFE and for any map of the form (1) the flux $F(m_j/n_j, K(\rho))$ satisfies the relation (8), and if the tail is constant the scaling (7) is also valid. In the case of the periodic tails, there are $s=$ (*period of the* CFE *tail*) subsequences of M_j/n_j such that the scaling (7) is verified for each of them with the same β. The exponents β and α depend only on the tail of ρ. This conjecture is consistent with the renormalization theory for such dynamical systems.

The scaling of the flux in the supercritical regime, i.e., over a sequence of periodic orbits that converge to a cantorus, has also been studied. We have observed that a transition between two different types of convergence parallels the transition from the weakly supercritical to the strongly supercritical regime. The weakly supercritical regime is characterized by two types of scalings: initial exponential decay with n_j of $F(m_j/n_j, k)$ and asymptotic, faster than exponential with n_j, approach the nonzero flux through the cantorus $F(\rho, k > K(\rho))$. In the strongly supercritical case the faster than exponential convergence, given by the relation (9), dominates. The coefficient λ in Eq. (9) is equal to the Lyapunov exponent of the corresponding cantorus.

Our computations are restricted to the examples of twist area-preserving maps known as the generalized standard maps. The assumption about the symmetry of the maps in the considered class was helpful in the computations of the periodic orbits, but does not seem to be fundamental, as is suggested by the renormalization arguments [23]. On the other hand, one does not expect the same type of scaling universality to be valid for the twist maps with a different type of criticality.

Another restriction is the type of quasiperiodic orbits that we have considered. All irrational rotation numbers that we have used had periodic tails of their continued fraction expansions, and more systematic computations have been done with numbers with a constant tail. Furthermore, due to computational difficulties, we were not able to perform systematic computations of the approximating periodic orbits for tails more than five. We cannot say anything about scaling near quasiperiodic orbits with Diophantine rotation numbers that are not quadratic irrationals.

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